

GROMOV-WITTEN THEORY OF TORIC BIRATIONAL TRANSFORMATIONS

PEDRO ACOSTA AND MARK SHOEMAKER

ABSTRACT. We investigate the effect of a general toric wall crossing on genus zero Gromov–Witten theory. Given two complete toric orbifolds X_+ and X_- related by wall crossing under variation of GIT, we prove that their respective I -functions are related by linear transformation and asymptotic expansion. We use this comparison to deduce a similar result for birational complete intersections in X_+ and X_- . This extends the work of the previous authors in [2] to the case of complete intersections in toric varieties, and generalizes some of the results of Coates–Iritani–Jiang [14] on the crepant transformation conjecture to the setting of non-zero discrepancy.

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1. INTRODUCTION

A longstanding goal in Gromov–Witten theory has been to understand how Gromov–Witten invariants change under birational transformation. Results in this direction have generally taken one of two forms. Namely, they focus on how a birational transformation affects either (a) the individual Gromov–Witten invariants of a space [18, 25, 24, 23, 27], or (b) certain formal structures defined via Gromov–Witten theory, e.g. quantum cohomology, generating functions, Lagrangian cones, etc... [30, 5, 6, 15, 13, 14, 35].

Results of the first type are often more general in terms of the cases they cover. The most systematic treatment along these lines of which we are aware is [23], which deals with wall crossing under general variation of GIT. Theorems of the second type are important however, as they show how certain Gromov–Witten-theoretic structures may be preserved under

birational transformation even as individual invariants are not. As an example of this phenomenon, in some instances the generating functions of Gromov–Witten invariants of birational spaces are related by analytic continuation [7]. These types of results are predicted by the *crepant transformation conjecture* [15, 16], but as such, deal almost exclusively with the case of K -equivalent birational spaces.

The purpose of this paper is to show that in fact analogous statements can be made even when the two spaces are not K -equivalent. In previous work [2], the authors conjectured that when X_+ and X_- are birational but not K -equivalent, the respective generating functions for genus zero Gromov–Witten invariants will be related by asymptotic expansion. This was verified in the particular case where the birational transformation was a blowup of $[\mathbb{C}^N/G]$ at the origin (for G a finite abelian group). The main result of the current paper verifies the above conjecture in much greater generality, showing that the same type of relation holds when X_+ and X_- are toric varieties (or orbifolds) related by variation of GIT as well as for compatible complete intersection varieties $Y_+ \subset X_+$ and $Y_- \subset X_-$.

1.1. Toric results. Let X_+ and X_- be complete toric orbifolds obtained as GIT quotients of a torus K acting on \mathbb{C}^m , and related by a wall crossing in the space of stability conditions. Under the assumption that X_+ and X_- are *not* K -equivalent, one can construct a common toric blow-up

$$\pi_{\pm} : \tilde{X} \rightarrow X_{\pm}$$

such that $\pi_+^*(K_{X_+}) - \pi_-^*(K_{X_-})$ is effective. (see Section 3.1).

By the work of Coates–Corti–Iritani–Tseng in [11], there exists an explicit expression for cohomology valued functions $I^+(y, z)$ and $I^-(\tilde{y}, z)$ which encode the genus zero Gromov–Witten theory of X_+ and X_- . Although not how they are defined a-priori, these I -functions can be understood as generating functions for the respective sets of Gromov–Witten invariants. Our main theorem relates these functions:

Theorem 1.1 (Theorem 5.7). *There exists a linear transformation*

$$L : H_{CR}^*(X_+)[z, z^{-1}] \rightarrow H_{CR}^*(X_-)[z, z^{-1}]$$

such that the power series asymptotic expansion of $L \cdot I^+(y, z)$ in a suitable variable (determined by the wall crossing) recovers the function $I^-(\tilde{y}, z)$.

The technical challenge in proving the above theorem is due to the fact that we lack an explicit expression for the asymptotic expansion of $I^+(y, z)$. We instead shift our focus from the functions I^+ and I^- themselves to the differential equations they satisfy. In particular, we construct a function $\mathbb{I}(y, z)$ taking values in $H_{CR}^*(X_-)[z, z^{-1}]$ which has asymptotic expansion equal to $I^-(\tilde{y}, z)$. We then show that the components of $\mathbb{I}(y, z)$ satisfy a system of differential equations, a completion of the GKZ system. The components of $I^+(y, z)$ give a *basis* of solutions to this system of equations, which then implies the existence of a linear map L sending $I^+(y, z)$ to $\mathbb{I}(y, z)$.

1.2. Complete intersections. Given a complete intersection $Y_+ \subset X_+$, the *quantum Lefschetz principal* (see [10] for a recent treatment) allows one to construct a function I_Y^+ encoding much of the Gromov–Witten theory of Y_+ . This function is expressed as an explicit term-by-term modification of the function I^+ given above.

In the final section we consider birational spaces Y_+ and Y_- , defined as complete intersections in X_+ and X_- respectively, and compatible with each other in a precise sense. We use Theorem 1.1 in combination with the *quantum Lefschetz principal* [10] to deduce a relationship between the generating functions I_Y^+ and I_Y^- .

Theorem 1.2 (Theorem 6.3). *There exists a linear transformation*

$$L : H_{\text{CR}}^*(X_+)[z, z^{-1}] \rightarrow H_{\text{CR}}^*(X_-)[z, z^{-1}]$$

such that the power series asymptotic expansion of $L \cdot I_Y^+(y, z)$ in a suitable variable (determined by the wall crossing) recovers the function $I_Y^-(\tilde{y}, z)$.

1.3. Relation to the crepant transformation conjecture. The crepant transformation conjecture has evolved over time in order to incorporate a successively broader range of cases, with the various incarnations usually differing in their viewpoint as to the central object of comparison. To place Theorem 1.1 in its proper context, we briefly recall some of the evolution of the conjecture. We will focus here on the genus zero picture.

An early form of the conjecture, due to Ruan [33], identifies the quantum cohomology of K -equivalent spaces X_+ and X_- after analytic continuation in the quantum parameters. Later a stronger conjecture, relating the full generating function of genus zero invariants, was given by Bryan–Graber [7] in the case of crepant resolutions of *hard Lefschetz orbifolds*. A similar statement has also been developed and proven for ordinary flops by Lee–Lin–Wang–et. al. [29, 27, 28]. Finally, the most relevant form of the conjecture for our purposes was proposed by Coates–Iritani–Tseng [15] using physical considerations from mirror symmetry. This version states that the overruled Lagrangian cones (see [21] for an introduction to Givental’s formalism) of X_+ and X_- should be identified after application of a linear symplectic transformation and suitable analytic continuation. This version of the conjecture is appealing in that not only does it relate the full set of genus zero Gromov–Witten invariants of X_+ and X_- , but also should apply in the most generality.

In the current work we choose to use the machinery of generating functions (the so-called *I*-functions) rather than Givental’s symplectic formalism, although either one fully determines the other (see [11]). This has the benefit of yielding a simple Bryan–Graber type correspondence at the level of *I*-functions, although a price is paid. Namely, extracting information about individual Gromov–Witten invariants from the *I*-function is often quite involved, although possible in principle—see Section 2.2.

A key ingredient in all three versions of the above conjecture is the analytic continuation of certain functions, and thus implicit in the conjecture is the statement that these functions are analytic in some variable. When X_+ and X_- are K -equivalent, $I^+(y, z)$ and $I^-(\tilde{y}, z)$ can often be shown to be analytic [14] but this fails for a general birational map $X_+ \dashrightarrow X_-$. In fact with the setup as in the previous section, I^+ will be analytic with an essential singularity at infinity and I^- will only be a formal power series, with radius of convergence zero.

We propose that in this case analytic continuation is to be replaced by a power series asymptotic expansion of I^+ , reflecting the inherent asymmetry of the map $X_+ \dashrightarrow X_-$. Furthermore the dimension of $H_{CR}^*(X_+)$ will be greater than that of $H_{CR}^*(X_-)$, and so the map L in Theorem 1.1 will not be invertible. The remarkable fact is that these appear to be the only modifications necessary to translate the crepant transformation conjecture into a statement which holds for much more general birational transformations (e.g. blow-ups).¹

Remark 1.3. We remark finally that a recent paper by Coates–Iritani–Jiang [14] gives yet another perspective on the crepant transformation conjecture, this time in terms of *quantum D -modules*. In [14], they prove a correspondence between quantum D -modules of K -equivalent toric orbifolds (and complete intersections) under variation of GIT. Our current paper is indebted to their work, as we use the same variation of GIT setup to conduct our toric wall crossing.

Remark 1.4. H. Iritani has also announced results comparing the genus zero Gromov–Witten invariants of $f : Y \rightarrow X$ when f is a toric blow-up or flip. His results are phrased in the language of quantum D -modules, and thus differ somewhat from our main theorem. It will be interesting to compare the two perspectives.

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2. GROMOV–WITTEN THEORY

In this section we recall those definitions and properties of Gromov–Witten theory which play a role in the correspondence. This section also serves to set notation. More details of the theory can be found in [1] or [17].

¹The existence of a non-invertible transformation L was already conjectured in [33] in this general context, although the role of asymptotic expansion had not yet been realized.

2.1. Invariants. Let X be a smooth, proper, Deligne–Mumford stack with projective coarse moduli space. Define the inertia stack IX as the fiber product

$$\begin{array}{ccc} IX & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

where Δ is the diagonal map. IX parametrizes pairs (x, g) where $x \in X$ and g is an element of the isotropy group of x . In the case where X is a stack quotient of the form $[V/G]$ with V a smooth variety and G an abelian group,

$$IX = \coprod_{g \in G} [V^g/G],$$

where V^g denotes the fixed locus of V with respect to g . There is a distinguished component corresponding to $g = e$ which is isomorphic to X . We call this the *untwisted sector*. The other components are called *twisted sectors*. In this paper we will be concerned with the case where G is an algebraic torus.

Notation 2.1. Unless otherwise stated, the coefficients of our cohomology groups are always in \mathbb{C} .

The relevant cohomology theory for use in orbifold Gromov–Witten theory is Chen–Ruan cohomology:

Definition 2.2 ([9, 8]). The *Chen–Ruan orbifold cohomology* of X is defined, as a vector space, as:

$$H_{CR}^*(X) := H^*(IX).$$

The inclusion of the untwisted sector $X \subset IX$ into the inertia stack allows us to view $H^*(X)$ as a summand of $H_{CR}^*(X)$.

There is a natural involution $inv : IX \rightarrow IX$ which sends the point (x, g) to (x, g^{-1}) . This endows the Chen–Ruan cohomology with a pairing given by

$$(\alpha, \beta)_{CR} := \int_{IX} \alpha \cup inv^*(\beta), \quad \alpha, \beta \in H_{CR}^*(X).$$

Fix a genus g , an integer $n \geq 0$, and a degree $d \in NE(X)_{\mathbb{Z}} = NE(X) \cap H_2(|X|; \mathbb{Z})$; there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X, d)$ parametrizing stable maps $f : C \rightarrow X$, where C is a pre-stable genus- g orbi-curve [1] with n marked points, and f is a representable stable map of degree d . The torus action on X induces an action on $\overline{\mathcal{M}}_{g,n}(X, d)$. Although gerbe structure prevents the global existence of evaluation maps $ev_i : \overline{\mathcal{M}}_{g,n}(X, d) \rightarrow IX$, in fact the pullback

$$ev_i : H_{CR}^*(X) \rightarrow H^*(\overline{\mathcal{M}}_{g,n}(X, d)),$$

can still be defined by identifying the cohomology of IX with that of \overline{IX} , the *rigidified inertia stack* (see [1] for details). For each marked point $1 \leq i \leq n$,

we define the i th ψ -class, $\psi_i \in H^*(\overline{\mathcal{M}}_{g,n}(X, d))$, to be the first Chern class of the i th universal cotangent line bundle L_i . Given classes $\alpha_1, \dots, \alpha_n \in H_{CR}^*(X)$ and integers $k_1, \dots, k_n \geq 0$, we define the Gromov–Witten invariant

$$\left\langle \alpha_1 \psi_1^{k_1}, \dots, \alpha_n \psi_n^{k_n} \right\rangle_{g,n,d}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{vir}} \bigcup_{i=1}^n ev_i^*(\alpha_i) \cup \psi_i^{k_i},$$

where $[\overline{\mathcal{M}}_{g,n}(X, d)]^{vir}$ denotes the *virtual fundamental class* on $\overline{\mathcal{M}}_{g,n}(X, d)$ [1].

2.2. Quantum cohomology and generating functions. Let $\mathbf{t} = \sum_{i \in I} t^i \phi_i$ represent a general point in $H_{CR}^*(X)$ for $\{\phi_i\}_{i \in I}$ a choice of basis. Given $\alpha_1, \dots, \alpha_n \in H_{CR}^*(X)$, and integers $k_1, \dots, k_n \geq 0$ as above, we define the double bracket

$$\left\langle \left\langle \psi_1^{a_1} \alpha_1, \dots, \psi_n^{a_n} \alpha_n \right\rangle \right\rangle^X := \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle \psi_1^{a_1} \alpha_1, \dots, \psi_n^{a_n} \alpha_n, \mathbf{t}, \dots, \mathbf{t} \right\rangle_{0,n+k,d}^X.$$

In general the sum on the right will converge only if we view the parameters $\{t^i\}_{i \in I}$ as formal variables. We denote $\mathbb{C}[[\{t^i\}_{i \in I}]]$ simply by $\mathbb{C}[[\mathbf{t}]]$.

Definition 2.3. The *quantum cohomology* of X ,

$$*_t : H_{CR}^*(X) \times H_{CR}^*(X) \rightarrow H_{CR}^*(X)[[\mathbf{t}]],$$

is defined by the equality

$$(\alpha *_t \beta, \gamma)_{CR} = \left\langle \left\langle \alpha, \beta, \gamma \right\rangle \right\rangle^X$$

for all $\alpha, \beta, \gamma \in H_{CR}^*(X)$.

As a consequence of the WDVV relations [1, 17], the operation $*_t$ endows $H_{CR}^*(X)[[\mathbf{t}]]$ with an associative product.

The *J-function*, a cohomology valued generating function of Gromov–Witten invariants of X first introduced by Givental [20], is useful in studying quantum cohomology.

Definition 2.4. The *J-function* of X is defined as

$$J^X(\mathbf{t}, z) = z + \mathbf{t} + \sum_{i \in I} \left\langle \left\langle \frac{\phi_i}{z - \psi_1} \right\rangle \right\rangle^X \phi^i$$

where $\{\phi^i\}_{i \in I}$ is the dual basis to $\{\phi_i\}_{i \in I}$ and $1/(z - \psi_1)$ denotes the corresponding expansion in $1/z$.

The *J-function* naturally takes values in $H_{CR}^*(X)[[\mathbf{t}]]((z^{-1}))$. We note that the quantum cohomology of X is fully determined by the *J-function*. The key point is that J^X satisfies the system of partial differential equations

$$z \frac{\partial}{\partial t^i} \frac{\partial}{\partial t^j} J^X(\mathbf{t}, z) = \sum_{k \in I} (\phi_i *_t \phi_j, \phi^k) \frac{\partial}{\partial t^k} J^X(\mathbf{t}, z),$$

which follows from the *topological recursion relations* [17]. The J -function can be upgraded to an endomorphism $\mathbf{J}^X(\mathbf{t}, z) : H_{CR}^*(X)((z^{-1}))[[\mathbf{t}]] \rightarrow H_{CR}^*(X)((z^{-1}))[[\mathbf{t}]]$ given by:

$$\mathbf{J}^X(\mathbf{t}, z) : Z(\mathbf{t}, z) \mapsto Z(\mathbf{t}, z) + \sum_{i \in I} \left\langle \left\langle \frac{\phi_i}{z - \psi_1}, Z(\mathbf{t}, z) \right\rangle \right\rangle^X \phi^i.$$

Via the *string equation* [1, 17], $z\mathbf{J}^X(\mathbf{t}, z)(1)$ can be seen to equal the original J -function. This motivates the following general definition:

Definition 2.5. Let q^1, \dots, q^r be formal parameters. An I -function for X is any cohomology-valued function of the form

$$(2.2.1) \quad I^X(\mathbf{q}, z) = z\mathbf{J}^X(\boldsymbol{\tau}(\mathbf{q}), z)(Z(\mathbf{q}, z)),$$

such that $Z(\mathbf{q}, z) \in H_{CR}^*(X)[z][[\mathbf{q}]]$ contains only *positive* powers of z . The map $\mathbf{q} \mapsto \boldsymbol{\tau}(\mathbf{q})$ is called the *mirror map*.

Definition 2.6. We say an I -function $I^X(\mathbf{q}, z)$ is *big* if there exist differential operators $\{P_i(z, z \frac{\partial}{\partial q^i})\}_{i \in I}$ which are polynomial in z and $z \frac{\partial}{\partial q^i}$ such that

$$(2.2.2) \quad z^{-1}P_i \left(I^X(\mathbf{q}, z) \right) = \phi_i + O(\mathbf{q}).$$

Using a form of Birkhoff factorization, one can prove (see e.g. [2, 14]):

Lemma 2.7. A big I -function $I^X(\mathbf{q}, z)$ explicitly determines the pullback of J^X under the mirror map.

As a consequence, if the image of the mirror map $\mathbf{q} \rightarrow \boldsymbol{\tau}(\mathbf{q})$ generates the Chen–Ruan cohomology of X , $I^X(\mathbf{q}, z)$ fully determines the quantum cohomology of X . In many cases an explicit description of the J -function is difficult to obtain, and it is more convenient to work with I -functions. This is the approach taken in this paper.

3. TORIC SETUP

We construct our toric varieties as stack quotients via GIT. The setup is almost identical to that in [14], so for the readers' convenience we use the same notation. The initial data consists of

- A torus $K \cong (\mathbb{C}^*)^r$;
- the lattice $\mathbb{L} = \text{Hom}(\mathbb{C}^*, K)$ of co-characters of K ;
- a set of characters $D_1, \dots, D_m \in \mathbb{L}^\vee = \text{Hom}(K, \mathbb{C}^*)$;
- a choice of a *stability condition* $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$.

Given the above, the map $(D_1, \dots, D_m) : K \rightarrow (\mathbb{C}^*)^m$ induces an action of K on \mathbb{C}^m . Given $I \subset \{1, \dots, m\}$, we denote by \bar{I} the complement of I . Define $\angle I$ to be the subset

$$\angle I := \left\{ \sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R}_{>0} \right\} \subset \mathbb{L}^\vee \otimes \mathbb{R},$$

and let $(\mathbb{C}^*)^I \times (\mathbb{C})^{\bar{I}} := \{(z_1, \dots, z_m) \mid z_i \neq 0 \text{ for } i \in I\}$.

Let $\mathcal{A}_\omega := \{I \subset \{1, \dots, m\} \mid \omega \in \angle I\}$. Consider the open set

$$U_\omega := \bigcup_{I \in \mathcal{A}_\omega} (\mathbb{C}^*)^I \times (\mathbb{C})^{\bar{I}}.$$

We define the toric stack

$$X_\omega := [U_\omega / K],$$

where brackets denote that we are taking the stack quotient.

From this perspective, $\mathbb{L}^\vee \otimes \mathbb{R}$ may be viewed as the set of stability parameters for the GIT quotient of \mathbb{C}^m by K . We restrict ourselves to ω in the non-negative span $\sum_{i=1}^m \mathbb{R}_{\geq 0} D_i$ so that X_ω is non-empty. This set acquires a wall and chamber structure as follows. For such $\omega \in \sum_{i=1}^m \mathbb{R}_{\geq 0} D_i$, define

$$C_\omega := \bigcap_{I \in \mathcal{A}_\omega} \angle I \subset \mathbb{L}^\vee \otimes \mathbb{R}.$$

Note that if $\omega' \in C_\omega$, X_ω and $X_{\omega'}$ are equal. Then the *chambers* of $\sum_{i=1}^m \mathbb{R}_{\geq 0} D_i$ are C_ω of maximal dimension r , and the *walls* are formed by higher codimension C_ω . If C_ω is of maximal dimension, X_ω will be a Deligne–Mumford stack. We also assume always that X_ω has proper coarse moduli space.

Fix a stability condition ω such that C_ω is of maximal dimension. The cohomology $H^*(X_\omega)$ is generated by the divisor classes $u_i := \{z_i = 0\}$ where z_i is the i th coordinate of $\mathbb{C}^m \supset U_\omega$, viewed as a homogeneous coordinate on X_ω . The cohomology of X_ω is computed explicitly in [5], although we will not require it here. There exists a linear map $\theta_\omega : \mathbb{L}^\vee \otimes \mathbb{R} \rightarrow H^2(X_\omega)$ defined such that

$$(3.0.1) \quad \theta_\omega(D_i) = u_i,$$

this is well defined by Equation 4.7 of [14].

For $f \in \mathbb{L} \otimes \mathbb{Q}$, define $I_f := \{i \in \{1, \dots, m\} \mid D_i \cdot f \in \mathbb{Z}\}$. Consider the set

$$\mathbb{K}_\omega := \{f \mid I_f \in \mathcal{A}_\omega\} \subset \mathbb{L} \otimes \mathbb{Q}.$$

Then $\mathbb{K}_\omega / \mathbb{L}$ indexes the fixed point components of IX_ω : $f \in \mathbb{K}_\omega / \mathbb{L}$ corresponds to the group element

$$\exp(f) := (\exp(2\pi i D_1 \cdot f), \dots, \exp(2\pi i D_r \cdot f)) \in K,$$

which in turn corresponds to the component $X_\omega^f := [U_\omega \cap (\mathbb{C}^m)^{\exp(f)} / K] \subset IX_\omega$. The Chen–Ruan orbifold cohomology of X_ω is then given as a graded vector space by

$$H_{CR}^*(X_\omega) = \bigoplus_{f \in \mathbb{K}/\mathbb{L}} H^*(X_\omega^f).$$

Let $\mathbb{1}_f$ denote the class in $H_{CR}^*(X_\omega)$ corresponding to the fundamental class of X_ω^f .

3.1. Wall crossing. Choose stability conditions $+$ and $-$ in $\mathbb{L}^\vee \otimes \mathbb{R}$ such that C_+ and C_- are of maximal dimension and are separated by a codimension one wall. Let W denote the hyperplane in $\mathbb{L}^\vee \otimes \mathbb{R}$ separating C_+ and C_- , and let $\overline{C_W} := W \cap \overline{C_+} = W \cap \overline{C_-}$ denote the (closure of the) corresponding wall. Let $e \in \mathbb{L}$ be a primitive generator of W^\perp . We assume that X_+ and X_- are *not* K -equivalent, or equivalently, that we can choose e such that $\sum_{j=1}^m D_j \cdot e > 0$. We may assume, after possibly switching the labeling of the stability conditions $+$ and $-$, that $\omega_+ \cdot e > 0$. One can construct (see Section 6.3 of [14]) a common toric blowup

$$(3.1.1) \quad \begin{array}{ccc} & \tilde{X} & \\ \pi_+ \swarrow & & \searrow \pi_- \\ X_+ & \xrightarrow{\phi} & X_- \end{array}$$

such that under our assumptions, $\pi_+^*(K_{X_+}) - \pi_-^*(K_{X_-})$ is effective.

Let $\tilde{\mathbb{L}}_\pm$ denote the lattice in $\mathbb{L} \otimes \mathbb{Q}$ generated by \mathbb{K}_\pm . In [14] it is proven that:

Lemma 3.1 ([14] Section 5.3). *There exist bases $\{p_1^+, \dots, p_r^+\}$ and $\{p_1^-, \dots, p_r^-\}$ of $\tilde{\mathbb{L}}_+^\vee$ and $\tilde{\mathbb{L}}_-^\vee$ respectively such that:*

- p_i^\pm lies in $\overline{C_\pm}$ for $1 \leq i \leq r$;
- $p_i^+ = p_i^- \in \overline{C_W}$ for $1 \leq i \leq r-1$;

For d in \mathbb{L} , resp. $\tilde{\mathbb{L}}_\pm$, let y^d denote the corresponding element in $\mathbb{C}[\mathbb{L}]$, resp. $\mathbb{C}[\tilde{\mathbb{L}}_\pm]$. We have an inclusion

$$\mathbb{C}[C_+^\vee \cap \mathbb{L}] \rightarrow \mathbb{C}[C_+^\vee \cap \tilde{\mathbb{L}}_+] \rightarrow \mathbb{C}[y_1, \dots, y_r]$$

given by $y^d \mapsto \prod_{i=1}^r y_i^{p_i^+ \cdot d}$. Similarly, the inclusion

$$\mathbb{C}[C_-^\vee \cap \mathbb{L}] \rightarrow \mathbb{C}[C_-^\vee \cap \tilde{\mathbb{L}}_-] \rightarrow \mathbb{C}[\tilde{y}_1, \dots, \tilde{y}_r]$$

is given by $y^d \mapsto \prod_{i=1}^r \tilde{y}_i^{p_i^- \cdot d}$.

The coordinates are related by the change of variables

$$(3.1.2) \quad \begin{aligned} \tilde{y}_i &= y_i y_r^{c_i} & 1 \leq i \leq r-1 \\ \tilde{y}_r &= y_r^{-c} \end{aligned}$$

where $c = -p_r^+ \cdot e / p_r^- \cdot e \in \mathbb{Q}_{>0}$ and $c_i \in \mathbb{Q}$ are determined by the change of basis from $\{p_i^+\}$ to $\{p_i^-\}$. Note that $p_r^+ \cdot e > 0$ while $p_r^- \cdot e < 0$.

3.2. The I -function. To simplify notation, for $k \in \mathbb{L} \otimes \mathbb{Q}$, let $y^k = \prod_{i=1}^r y_i^{p_i^+ \cdot k}$ and $\tilde{y}^k = \prod_{i=1}^r \tilde{y}_i^{p_i^- \cdot k}$. Note that $y^k = \tilde{y}^k$ under the change of variables (3.1.2), but we distinguish between the two when want to emphasize a particular coordinate system. Define

$$\sigma_\pm := \theta_\pm(p_i^\pm \log(y_i)),$$

where θ_\pm is as defined in (3.0.1). Define the functions $I^+(y, z)$ and $I^-(\tilde{y}, z)$ in $H_{CR}^*(X_+)[z, z^{-1}][\log y_1, \dots, \log y_r][[y_1, \dots, y_r]]$ and $H_{CR}^*(X_-)[z, z^{-1}][\log \tilde{y}_1, \dots, \log \tilde{y}_r][[\tilde{y}_1, \dots, \tilde{y}_r]]$ respectively by:

(3.2.1)

$$\begin{aligned} I^+(y, z) &:= z e^{\sigma_+/z} \sum_{k \in \mathbb{K}_+} y^k \left(\prod_{j=1}^m \frac{\prod_{a: \langle a \rangle = \langle D_j \cdot k \rangle, a \leq 0} (u_j + az)}{\prod_{a: \langle a \rangle = \langle D_j \cdot k \rangle, a \leq D_j \cdot k} (u_j + az)} \right) \mathbb{1}_{[-k]}, \\ I^-(\tilde{y}, z) &:= z e^{\sigma_-/z} \sum_{k \in \mathbb{K}_-} \tilde{y}^k \left(\prod_{j=1}^m \frac{\prod_{a: \langle a \rangle = \langle D_j \cdot k \rangle, a \leq 0} (u_j + az)}{\prod_{a: \langle a \rangle = \langle D_j \cdot k \rangle, a \leq D_j \cdot k} (u_j + az)} \right) \mathbb{1}_{[-k]}. \end{aligned}$$

For $k \in \mathbb{K}_\pm$, let I_k^\pm denote the corresponding summand:

$$\begin{aligned} (3.2.2) \quad I^+(y, z) &= z e^{\sigma_+/z} \sum_{k \in \mathbb{K}_+} y^k I_k^+, \\ I^-(\tilde{y}, z) &= z e^{\sigma_-/z} \sum_{k \in \mathbb{K}_-} \tilde{y}^k I_k^-. \end{aligned}$$

I^\pm fully determines the J -function J^\pm of X_\pm in the sense of Definition 2.6:

Theorem 3.2 ([11]). *I^\pm is a big I -function for X_\pm .*

Proof. This follows from the mirror theorem for toric stacks as given in [11] and is explicitly proven in Lemma 5.23 of [14]. \square

3.3. Differential equations. In this section we investigate the differential equations satisfied by I_\pm . What follows is just a small part of the larger theory of GKZ systems, which can be referenced in [19] or [3].

Since p_1^\pm, \dots, p_r^\pm forms a basis for $\mathbb{L}^\vee \otimes \mathbb{Q}$, each D_j can be written as a linear combination $D_j = \sum_{i=1}^r a_j^i p_i^+ = \sum_{i=1}^r \tilde{a}_j^i p_i^-$. Define the operator

$$\partial_j := \sum_{i=1}^r a_j^i y_i \frac{\partial}{\partial y_i} = \sum_{i=1}^r \tilde{a}_j^i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}.$$

This operator is designed so that

$$\partial_j y^d = (D_j \cdot d) y^d.$$

Furthermore, we observe that

$$\begin{aligned} \partial_j e^{\sigma_+/z} &= \partial_j \left(\prod_{i=1}^r y_i^{\theta_+(p_i^+)/z} \right) \\ &= \theta_+ \left(\sum_{i=1}^r a_j^i p_i^+ \right) / z \left(\prod_{i=1}^r y_i^{\theta_+(p_i^+)/z} \right) \\ &= (\theta_+(D_j) / z) e^{\sigma_+/z} \\ &= \frac{u_j}{z} e^{\sigma_+/z}. \end{aligned}$$

The same is true for the minus side: $\partial_j e^{\sigma_-/z} = \frac{u_j}{z} e^{\sigma_-/z}$.

Consider the GKZ system of differential operators $\{\Delta_d\}_{d \in \mathbb{L}}$

$$(3.3.1) \quad \Delta_d := \prod_{j|d_j > 0} \prod_{l=0}^{d_j-1} (z\partial_j - lz) - y^d \prod_{j|d_j < 0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz)$$

where $d_j := D_j \cdot d$. One can easily check the following, using the above facts about ∂_j .

Proposition 3.3. *The relation $\Delta_d I^+(y, z) = 0$ holds for all $d \in \mathbb{L}$. I^- satisfies the same set of equations after replacing y^d with \tilde{y}^d .*

3.4. Fractional calculus. It will be important in what follows to make use of so-called fractional derivatives [31]. In particular we would like to have a notion of $(d/dx)^a$ for $a \in \mathbb{Q}$. We note that for positive integers k and a ,

$$\left(\frac{d}{dx}\right)^a x^k = \frac{k!}{(k-a)!} x^{k-a}.$$

We can generalize this to $a \in \mathbb{C}$ as

$$\left(\frac{d}{dx}\right)^a x^k = \frac{\Gamma(k+1)}{\Gamma(k-a+1)} x^{k-a}.$$

To generalize this to other functions of x , we define, for $0 < a < 1$,

$$(3.4.1) \quad \left(\frac{d}{dx}\right)^a f(x) := \frac{1}{\Gamma(1-a)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^a} dt.$$

It is not hard to check that this agrees with the previous expression given by Gamma functions when $f(x) = x^k$. Then for arbitrary a ,

$$\left(\frac{d}{dx}\right)^a f(x) := \left(\frac{d}{dx}\right)^{\langle a \rangle} \circ \left(\left(\frac{d}{dx}\right)^{[a]} f(x) \right).$$

Remark 3.4. One must take care with the previous definition. Unlike the fractional *integral* (see, e.g., [31]), fractional derivatives unfortunately do not satisfy the semigroup property. That is, $\left(\frac{d}{dx}\right)^a \left(\frac{d}{dx}\right)^b f(x)$ does not necessarily equal $\left(\frac{d}{dx}\right)^b \left(\frac{d}{dx}\right)^a f(x)$.

4. REGULARIZATION AND WATSON'S LEMMA

In this section we construct a function $\mathbb{I} = \mathbb{I}(y_1, \dots, y_r, z)$ which is analytic in y_r and has an asymptotic expansion equal to I^- .

4.1. Regularization. We begin by noting the following fact:

Proposition 4.1. *I^+ is analytic with respect to y_r with radius of convergence ∞ .*

Proof. Fix $d \in \mathbb{K}_+$. We collect together all terms of $I^+(y, z)$ such that the exponent of y_i is $p_i^+ \cdot d$ for $1 \leq i \leq r-1$ and which are supported on $X_+^{[-d]}$. The corresponding function of y_r is (up to a factor of $ze^{\sigma_+/z} y^d$)

$$F_d^+(y_r, z) := \sum_{l \in \mathbb{Z}} y_r^{le} I_{d+le}^+.$$

Note that $y_r^{le} = y_r^{l(p_r^+ \cdot e)}$ as p_1^+, \dots, p_{r-1}^+ lie in e^\perp . For l sufficiently small, $I_{d+le}^+ = 0$, as $I^+(y, z)$ contains only positive powers of y_1, \dots, y_r by construction.

The ratio of the l th and $l+1$ st terms in $F_d(y_r z^{\sum e_j / p_r^+ \cdot e}, z)$ is

$$y_r^{p_r^+ \cdot e} \frac{\prod_{j|e_j < 0} \prod_{m=0}^{-e_j-1} (u_j/z + k_j + le_j - m)}{\prod_{j|e_j > 0} \prod_{m=1}^{e_j} (u_j/z + k_j + le_j + m)}.$$

Recall that $\sum e_j > 0$, thus the above expression (expanded in terms of u_j/z) goes to zero as $l \mapsto \infty$. Viewing z as a complex parameter, we conclude by the ratio test that $F_d(y_r z^{\sum e_j / p_r^+ \cdot e}, z)$ and therefore also $F_d(y_r, z)$ are convergent series in y_r with infinite radius of convergence. \square

We note that while I^+ is analytic in the y_r variable, I^- is not analytic in the \tilde{y}_r variable. To compare the two, we first *regularize* the function I^- by adjusting the coefficient of each \tilde{y}^k .

Define the *regularized* I^- function to be:

$$(4.1.1) \quad I_{reg}^-(\tilde{y}, z) := ze^{\sigma_-/z} \sum_{k \in \mathbb{K}_-} \frac{\tilde{y}^k I_k^-}{\Gamma(1 - (\sum e_j / p_r^- \cdot e)(p_r^- \cdot k + \theta_-(p_r^-)/z))}$$

Note that $I_k^- = 0$ for all k not in C_-^\vee . Because $p_r^- \cdot k \geq 0$ for all $k \in C_-^\vee$, the above modification is well defined. It will be convenient to make the change of variables $x = \tilde{y}_r^{-(p_r^- \cdot e / \sum e_j)}$ (recall that we've chosen e such that $p_r^- \cdot e < 0$). View $I_{reg}^- = I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z)$ as a function of the variables $\tilde{y}_1, \dots, \tilde{y}_{r-1}$ and x . In exactly the same manner as the previous proposition, I_{reg}^- can be shown to converge for x within a sufficiently small radius.

For future use we record the following, which holds for all $a > 0$ (see Section 3.4):

$$(4.1.2) \quad \begin{aligned} \left(\frac{\partial}{\partial x}\right)^a e^{\sigma_-/z} \tilde{y}^d &= \left(\frac{\partial}{\partial x}\right)^a \left(\left(\prod_{i=1}^{r-1} \tilde{y}_i^{\theta_-(p_i^-)/z + p_i^- \cdot d} \right) x^{-(\sum e_j / p_r^- \cdot e)(\theta_-(p_r^-)/z + p_r^- \cdot d)} \right) \\ &= \frac{\Gamma(1 - (\sum e_j / p_r^- \cdot e)(p_r^- \cdot d + \theta_-(p_r^-)/z))}{\Gamma(1 - (\sum e_j / p_r^- \cdot e)(p_r^- \cdot d + \theta_-(p_r^-)/z) - a)} \\ &\quad e^{\sigma_-/z} \left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) x^{-(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d) - a}. \end{aligned}$$

Proposition 4.2. *For all d in \mathbb{L} , I_{reg}^- satisfies the equation $\Delta_d^{reg} I_{reg}^- = 0$, where Δ_d^{reg} is defined to be*

$$\prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\partial_j - lz) - \left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) \left(\frac{\partial}{\partial x} \right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz)$$

if $p_r^- \cdot d < 0$, and

$$\left(\frac{\partial}{\partial x} \right)^{-(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\partial_j - lz) - \left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz)$$

if $p_r^- \cdot d \geq 0$.

Proof. The proof follows by a direct computation using the relations at the beginning of Section 3.3 along with (4.1.2). Although we are viewing I_{reg}^- as an analytic function in x , by the same argument as in the previous proposition it converges absolutely for x in a sufficiently small radius. Thus it suffices to check term-by-term using the power series expansion of (4.1.1), which we may re-write as $I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z) = \sum_{k \in \mathbb{K}_-} I_{reg,k}^-$ where

$$I_{reg,k}^- = ze^{\sigma_- / z} \left(\prod_{j=1}^m \frac{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq 0} (u_j + az)}{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq k_j} (u_j + az)} \right) \frac{\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot k} x^{-(\sum e_j / p_r^- \cdot e)(p_r^- \cdot k)}}{\Gamma(1 + (\sum e_j / p_r^- \cdot e)(p_r^- \cdot k + \theta_-(p_r^-))} \mathbb{1}_{[-k]}.$$

Recall that $I_{reg,k}^-$ is 0 unless $k \in C_-^\vee$. Fix such a $k \in \mathbb{K}_-$. Assume that $p_r^- \cdot d < 0$. Applying $\prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\partial_j - lz)$ to $I_{reg,k}^-$ amounts to multiplying by

$$\prod_{j|d_j \geq 0} \prod_{l=0}^{d_j-1} (u_j + k_j z - lz),$$

which yields

(4.1.3)

$$ze^{\sigma_- / z} \left(\prod_{j|d_j>0} \frac{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq 0} (u_j + az)}{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq k_j - d_j} (u_j + az)} \right) \left(\prod_{j|d_j<0} \frac{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq 0} (u_j + az)}{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq k_j} (u_j + az)} \right) \cdot \frac{\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot k} x^{-(\sum e_j / p_r^- \cdot e)(p_r^- \cdot k)}}{\Gamma(1 + (\sum e_j / p_r^- \cdot e)(p_r^- \cdot k + \theta_-(p_r^-))} \mathbb{1}_{[-k]}.$$

Note that the above expression is 0 if $k - d \notin C_-^\vee$, since it is supported on the subspace of $X_+^{[-k]}$ given by $z_j = 0$ for all j such that $k_j - d_j \in \mathbb{Z}_{<0}$. On the other hand, if $k - d \in C_-^\vee$, by applying

$$\left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) \left(\frac{\partial}{\partial x} \right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz)$$

to $I_{reg,k-d}^-$ we obtain

(4.1.4)

$$\left(\prod_{j|d_j>0} \frac{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq 0} (u_j + az)}{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq k_j - d_j} (u_j + az)} \right) \left(\prod_{j|d_j<0} \frac{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq 0} (u_j + az)}{\prod_{a:\langle a \rangle = \langle k_j \rangle, a \leq k_j} (u_j + az)} \right) \cdot$$

$$\left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) \left(\frac{\partial}{\partial x} \right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} z e^{\sigma_- / z} \frac{\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot (k-d)} x^{-(\sum e_j / p_r^- \cdot e)(p_r^- \cdot (k-d))}}{\Gamma(1 + (\sum e_j / p_r^- \cdot e)(p_r^- \cdot (k-d) + \theta_-(p_r^-)))} \mathbb{1}_{[-k]}.$$

By (4.1.2) this is seen to equal (4.1.3). The calculation for $p_r^- \cdot d > 0$ is similar. \square

Proposition 4.3. *The regularized I-function $I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z)$ is analytic with respect to x , and can be analytically continued to all but a finite number of points in the region $\arg(x) < \pi$. Furthermore, after fixing powers of y_1, \dots, y_{r-1} , the corresponding coefficient of $I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z)$ is bounded by an algebraic function as $x \mapsto \infty$.*

Proof. Similarly to Proposition 4.1, we fix $d \in \mathbb{K}_-$, and group together the terms of $I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z)$ corresponding to degree $d - le$ for $l \in \mathbb{Z}$:

$$F_{reg,d}^-(x, z) := \sum_{l \in \mathbb{Z}} I_{reg,d-le}^-.$$

Again $I_{reg,d-le}^- = 0$ for sufficiently small l .

Consider the differential operator Δ_e^{reg} from the above proposition:

$$\prod_{j|e_j>0} \prod_{l=0}^{e_j-1} \left(\frac{-(p_r^- \cdot e)e_j}{\sum_m e_m} x \frac{\partial}{\partial x} + d_j + u_j - l \right) -$$

$$\left(\frac{\partial}{\partial x} \right)^{\sum e_j} \prod_{j|e_j<0} \prod_{l=0}^{-e_j-1} \left(\frac{-(p_r^- \cdot e)e_j}{\sum_m e_m} x \frac{\partial}{\partial x} + d_j + u_j - l \right)$$

we see that this annihilates $F_{reg,d}^-(x, z)$. The corresponding differential equation has singularities at $0, \infty$, and all x satisfying $\left(\frac{-p_r^- \cdot e}{\sum e_j} x \right)^{\sum e_j} = \prod_{\{j|e_j \neq 0\}} e_j^{-e_j}$. The singularity at $x = \infty$ is regular, which implies that for each $d \in \mathbb{K}_-$, $F_{reg,d}^-(x, z)$ is bounded by an algebraic function as $x \mapsto \infty$. \square

4.2. Laplace transform and Watson's lemma. Define a new function $\mathbb{I}(\tilde{y}_1, \dots, \tilde{y}_{r-1}, u, z)$ using the Laplace transform:

$$\begin{aligned}
(4.2.1) \quad \mathbb{I}(\tilde{y}_1, \dots, \tilde{y}_{r-1}, u, z) &:= u \mathcal{L} \left(I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z) \right) \\
&= u \int_0^\infty e^{-ux} I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z) dx.
\end{aligned}$$

Here again we are viewing I_{reg}^- as an analytic function in x . We choose our path of integration to be any ray from 0 to ∞ with

$$0 < \arg(x) < \min(\pi/2, \pi / \sum e_j)$$

to avoid the singular points found in Proposition 4.3. By Proposition 4.3, after fixing powers of $\tilde{y}_1, \dots, \tilde{y}_{r-1}$, the corresponding coefficient of $I_{reg}^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, x, z)$ is in $O(x^N)$ for some $N > 0$ as $x \mapsto \infty$. Thus the integral is well defined for all u such that $\Re(u) > 0$. In fact the path integral does not depend on the choice of ray as long as $\arg(x)$ is within the range given above. The function $\mathbb{I}(\tilde{y}_1, \dots, \tilde{y}_{r-1}, u, z)$ is defined for all u such that $|\arg(u)| < \pi/2$.

Proposition 4.4. *The asymptotic expansion of \mathbb{I} is given by I_- :*

$$\mathbb{I}(\tilde{y}_1, \dots, \tilde{y}_{r-1}, u, z) \sim I^-(\tilde{y}_1, \dots, \tilde{y}_{r-1}, \tilde{y}_r, z) \Big|_{\tilde{y}_r = u^{(\sum e_j / p_r^- \cdot e)}}$$

as $u \mapsto \infty$ along any ray satisfying $|\arg(u)| < \pi/2$.

Proof. First, view the classes u_i ($1 \leq i \leq m$) as formal parameters. In this case the corresponding asymptotic expansion is a direct application of Watson's Lemma. One version of Watson's Lemma states that given a function $\phi(x) = x^\lambda g(x)$ where:

- $g(x)$ is infinitely differentiable at $x = 0$ and $g(0) \neq 0$;
- $\Re(\lambda) > -1$;
- there exists a $b > 0$ such that $|\phi(x)| < e^{bx}$ for $|x|$ sufficiently large;

then the following asymptotic expansion holds:

$$u \int_0^\infty e^{-ux} \phi(x) dx \sim \sum_{n=0}^\infty \frac{g^{(n)}(0) \Gamma(1 + \lambda + n)}{n! u^{\lambda+n}}$$

as $u \mapsto \infty$. One proof of this lemma is given in [32], Proposition 2.1. Although the conditions on $\phi(x)$ are slightly different in [32] than those stated above in fact the same proof implies the result in this case as well (see [32, Section 2.3] or [34] for the version given above).

To apply the lemma, group the terms of I^- according to their powers of x modulo $\sum e_j$. Write I^- as a sum of $N = -p_r^- \cdot e$ functions, $L_0, \dots, L_{-p_r^- \cdot e - 1}$, where the function L_l contains those terms of I^- with x -powers of the form $x^{(\sum e_j / p_r^- \cdot e)(\theta_-(p_r^-) / z + l) + k \sum e_j}$ for k in \mathbb{Z} . In other words, $L_l(x, z)$ can be written in the form $x^{(\sum e_j / p_r^- \cdot e)(\theta_-(p_r^-) / z + l)} g(x)$ where $g(x)$ is a power series in x and $g(0) \neq 0$. Let $L_{l, reg}$ denote the corresponding summand of I_{reg}^- and let $\mathbb{I}_l := u \mathcal{L}(L_{l, reg}(x, z))$. Watson's Lemma implies that $\mathbb{I}_l \sim L_l$ for each $0 \leq$

$l < -p_r^- \cdot e - 1$. Summing over l then gives the result when the cohomology classes u_i are viewed as formal parameters.

To then prove the statement in $H_{CR}^*(X_-)$, we use the above, and note that for each $k \in \mathbb{K}_-/\mathbb{L}$, $H^*(X_-^{[-k]})$ is a quotient of the algebra $\mathbb{C}[u_1, \dots, u_n]$ (see e.g. [5]). \square

Proposition 4.5. $\mathbb{I}(\tilde{y}_1, \dots, \tilde{y}_{r-1}, u, z)$ satisfies the differential equation (4.2.2)

$$\prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\bar{\partial}_j - lz) \mathbb{I} - \left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) u^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\bar{\partial}_j - lz) \mathbb{I} = 0,$$

where

$$\bar{\partial}_j := \sum_{i=1}^{r-1} \tilde{a}_j^i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i} + \tilde{a}_r^j \frac{p_r^- \cdot e}{\sum e_j} u \frac{\partial}{\partial u}.$$

Proof. The proof follows by applying $u\mathcal{L}$ to the equation $\Delta_d^{reg} I_{reg}^- = 0$ of Proposition 4.2 and using various well-known properties of the Laplace transform.

We recall the following facts about the Laplace transform. First, if $f(x)$ and $g(x)$ are functions with well defined Laplace transforms and $f(0) = 0$, then:

- (1) $\mathcal{L}(xf(x)) = -\frac{d}{du} \mathcal{L}(f(x));$
- (2) $\mathcal{L}\left(\frac{d}{dx} f(x)\right) = u \mathcal{L}(f(x));$
- (3) $\mathcal{L}((f * g)(x)) = \mathcal{L}(f(x)) \mathcal{L}(g(x));$

where $(f * g)(x)$ denotes the convolution of f and g . Let $F(u) = u\mathcal{L}(f(x))$, then properties (1) and (2) together imply that

$$(4.2.3) \quad u\mathcal{L}\left(x \frac{d}{dx} f(x)\right) = -u \frac{d}{du} F(u).$$

Furthermore, by (1), (2), and (3) above, together with definition (3.4.1), one can check that for any $a \in \mathbb{Q}$,

$$(4.2.4) \quad u\mathcal{L}\left(\left(\frac{d}{dx}\right)^a f(x)\right) = u^a F(x),$$

provided that the left hand side is well defined and that $\left(\frac{d}{dx}\right)^{a-n} f(x)|_{x=0} = 0$ for all integers $1 \leq n \leq \lfloor a+1 \rfloor$.

It is not *a priori* obvious that (4.2.3) holds when applied to I_{reg}^- , and in fact the left hand side of (4.2.4) will not always be defined for $f(x)$ equal to I_{reg}^- . Nonetheless we will show that equations (4.2.3) and (4.2.4) hold true in the cases of relevance to us.

We first check (4.2.3). One can use the power series expansion (4.1.1), together with the fact that

$$(4.2.5) \quad u\mathcal{L}\left(x^{\lambda+a}\right) = \Gamma(1+a+\lambda)u^{-\lambda-a},$$

where both the right and left hand side are meant to represent power series expansions in λ , for λ a formal parameter and $a \geq 0$. This implies that

$$(4.2.6) \quad u\mathcal{L} \prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\partial_j - lz) I_{reg}^- = \prod_{j|d_j>0} \prod_{l=0}^{d_j-1} (z\bar{\partial}_j - lz) u\mathcal{L} \left(I_{reg}^- \right).$$

Checking (4.2.4) is slightly more subtle, since in general the Laplace transform of $\left(\frac{d}{dx}\right)^a I_{-,reg}(x)$ is not defined. However consider, for instance, the operator

$$\left(\prod_{i=1}^{r-1} y_i^{p_i^- \cdot d} \right) \left(\frac{\partial}{\partial x} \right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz),$$

coming from the right hand side of Δ_d^{reg} , for some $d \in \mathbb{L}$ such that $p_r^- \cdot d < 0$. From equation (4.1.4), one sees that $\prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz) I_{reg,k-d}^-$ vanishes unless $k \in C_-^\vee$. This implies that $\prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz) I_{reg,k}^- = 0$ for all k such that $p_r^- \cdot d > p_r^- \cdot k$. We deduce that the expression

$$\prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz) I_{reg}^-(x, z)$$

can be expanded as $e^{\sigma_- / z} x^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} f(x, z)$, where $f(x, z)$ is a power series in $x^{-1/p_r^- \cdot e}$ with positive radius of convergence. Applying the operator $\left(\frac{\partial}{\partial x}\right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)}$, the result takes the form $e^{\sigma_- / z} g(x, z)$ for $g(x, z)$ a power series in $x^{-1/p_r^- \cdot e}$. This has a well defined Laplace transform. Using equation (4.2.4), one can then conclude that

$$(4.2.7) \quad \begin{aligned} & u\mathcal{L} \left(\left(\frac{\partial}{\partial x} \right)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz) I_{reg}^- \right) \\ &= u^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \cdot u\mathcal{L} \left(\prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\partial_j - lz) I_{reg}^- \right) \\ &= u^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \prod_{j|d_j<0} \prod_{l=0}^{-d_j-1} (z\bar{\partial}_j - lz) u\mathcal{L} \left(I_{reg}^- \right). \end{aligned}$$

Equations (4.2.6) and (4.2.7) together imply (4.2.2) in the case where $p_r^- \cdot d < 0$. The other case is similar. \square

Consider now the change of variables

$$(4.2.8) \quad \begin{aligned} \tilde{y}_i &= y_i y_r^{c_i} & 1 \leq i \leq r-1 \\ u &= y_r^{-c p_r^- \cdot e / \sum e_j}. \end{aligned}$$

View the function $\mathbb{I}(y_1, \dots, y_{r-1}, y_r, z)$ as a function of the variables y_1, \dots, y_r . Note that under this change of variables,

$$\left(\prod_{i=1}^{r-1} \tilde{y}_i^{p_i^- \cdot d} \right) (u)^{(\sum e_j / p_r^- \cdot e)(p_r^- \cdot d)} \mapsto \left(\prod_{i=1}^{r-1} y_i^{p_i^- \cdot d} y_r^{c_i p_i^- \cdot d} \right) y_r^{-c(p_r^- \cdot d)} = \prod_{i=1}^r y_i^{p_i^+ \cdot d} = y^d.$$

Similarly,

$$\bar{\partial}_j = \sum_{i=1}^{r-1} \tilde{a}_j^i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i} + \tilde{a}_r^j \frac{p_r^- \cdot e}{\sum e_j} u \frac{\partial}{\partial u} \mapsto \sum_{i=1}^r a_j^i y_i \frac{\partial}{\partial y_i} = \partial_j$$

Under this change of variables, the previous proposition states the following.

Proposition 4.6. *For all $d \in \mathbb{L}$, $\mathbb{I}(y_1, \dots, y_r, z)$ satisfies the equation $\Delta_d \mathbb{I}(y, z) = 0$, where Δ_d is defined as in (3.3.1).*

5. THE CORRESPONDENCE

From the previous section we know that $\mathbb{I}(y, z)$ has asymptotic expansion equal to $I^-(\tilde{y}, z)$ as $y_r \mapsto \infty$ along a suitable ray. In this section we deduce the main theorem by proving the existence of a linear transformation $L : H_{CR}^*(X_+)[z, z^{-1}] \rightarrow H_{CR}^*(X_-)[z, z^{-1}]$ which maps $I^+(y, z)$ to $\mathbb{I}(y, z)$.

5.1. The weak Fano case. For simplicity, we first consider the case where X_+ is extended weak Fano.

Definition 5.1. The Deligne–Mumford stack X is said to be *weak Fano* if $\rho = c_1(TX)$ is in the closure of the Kähler cone.

Given a toric stack $X_\omega = [U_\omega / K]$ defined via the characters $D_1, \dots, D_m \in \mathbb{L}^\vee$ and the stability condition $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$, we say X_ω is *extended weak Fano* if

$$\hat{\rho} := \sum_{i=1}^m D_i \in \overline{C_\omega}.$$

The notion of extended weak Fano depends upon the choice of presentation of X_ω as a toric quotient, however it is only slightly stronger than weak Fano.

The reason to consider this case is due to a simplification arising from the following result of Iritani.

Proposition 5.2. *When $\hat{\rho} \in \overline{C_+}$, the components of $I^+(y, z)$ give a basis of solutions to the GKZ system of equations $\{\Delta_d f(y, z) = 0\}_{d \in \mathbb{L}}$, where Δ_d is defined in (3.3.1).*

Proof. This follows immediately from Lemma 3.9 and Proposition 4.4 of [26]. \square

We immediately deduce the main result in the case where X_+ is extended weak Fano.

Theorem 5.3. *There exists a linear transformation $L : H_{CR}^*(X_+)[z, z^{-1}] \rightarrow H_{CR}^*(X_-)[z, z^{-1}]$ such that $L \cdot I^+(y, z) = \mathbb{I}(y, z) \sim I^-(\tilde{y}, z)$ as $y_r \mapsto \infty$ along any ray with $|\arg(y_r)|$ sufficiently close to 0.*

Proof. By Proposition 4.6, the components of $\mathbb{I}(y, z)$ give solutions to the GKZ system of equations. By the previous proposition, each component of $\mathbb{I}(y, z)$ must therefore be a linear combination of the components of $I^+(y, z)$. This uniquely defines the map $L : H_{CR}^*(X_+)[z, z^{-1}] \rightarrow H_{CR}^*(X_-)[z, z^{-1}]$.

The second claim, that $\mathbb{I}(y, z) \sim I^-(\tilde{y}, z)$ as $y_r \mapsto \infty$, is Proposition 4.4. See (4.2.8) for the change of variables between u and y_r . \square

5.2. The general case. When X_+ is not extended weak Fano, the dimension of the GKZ system is larger than $\dim(H_{CR}^*(X_+))$, and so the I -function I^+ no longer generates all solutions. In this more general setting, one must consider the completion of the corresponding D -module at $y_1 = \dots = y_r = 0$ to recover the full set of differential equations satisfied by I^+ . The proof closely follows the argument of Iritani given in Section 4.2 of [26], but with the crucial adjustment to the non-weak Fano case provided by a dimension bound from Gonzalez–Woodward [22].

Notation 5.4. To simplify the expressions to come, we will use the following notational substitution when defining various rings and modules: $y := y_1, \dots, y_r$; $y^{-1} := y_1^{-1}, \dots, y_r^{-1}$; $\frac{\partial}{\partial y} := \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_r}$ etc... Let D denote the ring of differential operators on $\mathbb{C}[y]$ with the usual relations.

Let \mathcal{I}_{GKZ} denote the ideal in $\mathbb{C}[z][y, y^{-1}]\langle zy \frac{\partial}{\partial y} \rangle$ generated by the differential operators Δ_d of (3.3.1). Define the D -module $M_{GKZ} := \mathbb{C}[z][y, y^{-1}]\langle zy \frac{\partial}{\partial y} \rangle / \mathcal{I}_{GKZ}$.

Let \mathcal{I}_{poly} denote the intersection $\mathcal{I}_{GKZ} \cap \mathbb{C}[z][y]\langle zy \frac{\partial}{\partial y} \rangle$ and consider the corresponding D -module $M_{poly} := \mathbb{C}[z][y]\langle zy \frac{\partial}{\partial y} \rangle / \mathcal{I}_{poly}$. Let \widehat{M}_{poly} denote $\mathbb{C}[z][[y]]\langle zy \frac{\partial}{\partial y} \rangle / \overline{\mathcal{I}_{poly}}$, where $\overline{\mathcal{I}_{poly}}$ is the closure of \mathcal{I}_{poly} in $\mathbb{C}[z][[y]]\langle zy \frac{\partial}{\partial y} \rangle$ in the y -adic topology.

By Proposition 4.4 of [26], there exists a Zariski open subset \mathcal{M}° of $\text{Spec } \mathbb{C}[y, y^{-1}]$ on which $M_{GKZ}^\circ := \mathcal{O}_{\mathcal{M}^\circ}[z] \otimes_{\mathbb{C}[z, y, y^{-1}]} M_{GKZ}$ is finitely generated as an $\mathcal{O}_{\mathcal{M}^\circ}[z]$ -module. By Lemma 3.8 of [26], we may assume, after possibly shrinking \mathcal{M}° , that $\text{Spec } \mathbb{C}[y] \setminus \mathcal{M}^\circ$ is a union of hypersurfaces $V' \cup V''$, where every irreducible component of V' avoids the origin, and V'' is given by the union of the coordinate hyperplanes $\{y_1 \cdots y_r = 0\}$. Then $\mathcal{M}^\circ = \mathcal{M}' \cap \text{Spec } \mathbb{C}[y, y^{-1}]$, where $\mathcal{M}' = \mathcal{M} \setminus V'$. Note that $\widehat{\Gamma(\mathcal{O}_{\mathcal{M}'})}$, the ring of formal functions on \mathcal{M}' in a neighborhood of the origin, is equal to $\widehat{\mathbb{C}[y]} = \mathbb{C}[[y]]$.

Proposition 5.5. *The D -module $\widehat{M}_{poly}^\circ := \widehat{M}_{poly}[y^{-1}]$ is a coherent sheaf of rank at most $\dim(H_{CR}^*(X_+))$ over $\text{Spec}(\mathbb{C}[z][[y]][y^{-1}])$.*

Proof. M_{GKZ}° is finitely generated as an $\mathcal{O}_{\mathcal{M}^\circ}$ -module. Note that M_{GKZ}° is equal to $M_{poly}^\circ := \mathcal{O}_{\mathcal{M}^\circ}[z] \otimes_{\mathbb{C}[z,y]} M_{poly}$, since M_{GKZ} equals M_{poly} over $\text{Spec } \mathbb{C}[y, y^{-1}] \supset \mathcal{M}^\circ$. Thus M_{poly}° is finitely generated as an $\mathcal{O}_{\mathcal{M}^\circ}[z]$ -module, and therefore $\mathbb{C}[z][[y]] \otimes_{\mathcal{O}_{\mathcal{M}'[z]}} M_{poly}^\circ$ is finitely generated as a $\mathbb{C}[z][[y]] \otimes_{\mathcal{O}_{\mathcal{M}'[z]}} \mathcal{O}_{\mathcal{M}^\circ}[z]$ -module. By the surjection $\mathbb{C}[z][[y]] \otimes_{\mathcal{O}_{\mathcal{M}'[z]}} M_{poly}^\circ \rightarrow \widehat{M}_{poly}^\circ$ we conclude that \widehat{M}_{poly}° is finitely generated as a $\mathbb{C}[z][[y]] \otimes_{\mathcal{O}_{\mathcal{M}'[z]}} \mathcal{O}_{\mathcal{M}^\circ}[z] = \mathbb{C}[z][[y]][y^{-1}]$ -module.

Thus \widehat{M}_{poly}° is a coherent sheaf over $\text{Spec}(\mathbb{C}[z][[y]][y^{-1}])$. \widehat{M}_{poly}° is also endowed with a flat connection away from $z = 0$ as described in section 4.2 of [26]. Therefore \widehat{M}_{poly}° is locally free away from $z = 0$. The restriction to $z = 0$ of \widehat{M}_{poly}° is isomorphic to the completion of the Batyrev ring: $\mathbb{C}[[y]][y^{-1}] \otimes_{\mathbb{C}[y]} B(X_+)$ (see [4] or [22] for the definition of the Batyrev ring $B(X_+)$). By Theorem 4.23 of Gonzalez-Woodward [22], this has rank equal to $\dim(H_{CR}^*(X_+))$. By Nakayama's lemma, the fiber over every point of $\text{Spec}(\mathbb{C}[z][[y]][y^{-1}])$ has dimension at most $\dim(H_{CR}^*(X_+))$ \square

Due to the presence of powers of $\log(y_j)/z$ in the I -function $I^+(y, z)$, we must extend scalars in order to compare $I^+(y, z)$ to the GKZ D -module. Define

$$\widehat{M}_{poly}^\circ[\log(y)] := \mathbb{C}[y, z, 1/z, \log(y)] \otimes_{\mathbb{C}[y,z]} \widehat{M}_{poly}^\circ.$$

Then $\widehat{M}_{poly}^\circ[\log(y)]$ is a coherent sheaf of rank at most $\dim(H_{CR}^*(X_+))$ over $\text{Spec}(\mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)])$. $\widehat{M}_{poly}^\circ[\log(y)]$ also acquires the structure of a D -module by defining

$$\frac{\partial}{\partial y_j} \log(y_m) := \frac{\delta_{j,m}}{y_m}.$$

Define a map of $\mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)] \otimes_{\mathbb{C}[y]}$ D -modules

$$L^+ : \widehat{M}_{poly}^\circ[\log(y)] \rightarrow H_{CR}^*(X_+) \otimes \mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)]$$

by sending $P(z, y, \log(y), zy \frac{\partial}{\partial y}) \mapsto P(z, y, \log(y), zy \frac{\partial}{\partial y}) (I^+(y, z))$. Since I^+ satisfies $P \cdot (I^+) = 0$ for all $P \in \mathcal{I}_{poly}$, it follows by continuity that I^+ is annihilated by all of $\overline{\mathcal{I}_{poly}}$. Since I^+ is a big I -function (see (2.2.2)), the map L^+ is surjective. Comparing ranks by the above proposition we conclude the following:

Corollary 5.6. *The map*

$$L^+ : \widehat{M}_{poly}^\circ[\log(y)] \rightarrow H_{CR}^*(X_+) \otimes \mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)]$$

defined above is an isomorphism of $\mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)] \otimes_{\mathbb{C}[y]}$ D -modules.

We arrive at our main result.

Theorem 5.7. *There exists a linear transformation $L : H_{CR}^*(X_+)[z, z^{-1}] \rightarrow H_{CR}^*(X_-)[z, z^{-1}]$ such that $L \cdot I^+(y, z) = \mathbb{I}(y, z) \sim I^-(\tilde{y}, z)$ as $y_r \mapsto \infty$ along a ray with $|\arg(y_r)|$ sufficiently close to 0.*

Proof. By the same argument as in the previous corollary, there is a well-defined map of $\mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)] \otimes_{\mathbb{C}[y]} D$ -modules

$$L^- : \widehat{M}_{poly}^\circ[\log(y)] \rightarrow H_{CR}^*(X_-) \otimes \mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)]$$

defined by $P(z, y, \log(y), zy \frac{\partial}{\partial y}) \mapsto P(z, y, \log(y), zy \frac{\partial}{\partial y})(\mathbb{I}(y, z))$. Define L to be the composition $L^- \circ (L^+)^{-1}$, viewed as a map of modules (rather than sheaves). Since both L^+ and L^- are maps of D -modules, $\frac{\partial}{\partial y_j} L^\pm = L^\pm \frac{\partial}{\partial y_j}$ for $1 \leq j \leq r$. One can check that a function $f(y, \log(y), z) \in \mathbb{C}[z, 1/z][[y]][y^{-1}, \log(y)]$ is constant if and only if $\frac{\partial}{\partial y_j} f(y, \log(y), z) = 0$ for $1 \leq j \leq r$. We conclude that L^+ and L^- are both $\mathbb{C}[z, z^{-1}]$ -linear maps, and therefore so is L .

The second claim, that $\mathbb{I}(y, z) \sim I^-(\tilde{y}, z)$ as $y_r \mapsto \infty$, is Proposition 4.4. \square

6. COMPLETE INTERSECTIONS

In this section we use the above results combined with the quantum Lefschetz principle to deduce corresponding statements for certain birational complete intersections $Y_+ \subset X_+$ and $Y_- \subset X_-$.

Given $\phi : X_+ \dashrightarrow X_-$ as in Section 3.1 and a choice of characters $E_1, \dots, E_n \in \mathbb{L}^\vee$ (see Section 3 for notation), one obtains corresponding line bundles $L_{1,\pm}, \dots, L_{n,\pm}$ over X_\pm . Let $E_+ = \oplus_j L_{j,+} \rightarrow X_+$ and $E_- = \oplus_j L_{j,-} \rightarrow X_-$ denote vector bundles obtained as the direct sum of the $+/-$ line bundles. Let s_+ and s_- be regular sections of E_+ and E_- respectively, and let $Y_+ \subset X_+$ and $Y_- \subset X_-$ denote the corresponding substacks.

Following [14], we require that:

- (1) For $1 \leq j \leq n$, E_j lies in $\overline{\mathcal{C}}_W$;
- (2) For $1 \leq j \leq n$, $L_{j,\pm}$ is pulled back from the coarse space $|X_\pm|$;
- (3) s_+ and s_- are compatible with the map $\phi : X_+ \dashrightarrow X_-$;

In the above situation, ϕ induces a birational map $\phi_Y : Y_+ \dashrightarrow Y_-$.

For $1 \leq j \leq n$, let $v_j \in H^*(X_\pm)$ denote $c_1(L_{j,\pm})$ (by abuse of notation we will let v_j denote the cohomology class in either X_+ or X_-). We define the

E_{\pm} -twisted I -function on X_{\pm} as

$$(6.0.1) \quad \begin{aligned} I_{tw}^+(y, z) &:= ze^{\sigma_+/z} \sum_{k \in \mathbb{K}_+} y^k I_k^+ \left(\prod_{j=1}^n \prod_{a=1}^{E_j \cdot k} (v_j + az) \right) \\ I_{tw}^-(\tilde{y}, z) &:= ze^{\sigma_-/z} \sum_{k \in \mathbb{K}_-} \tilde{y}^k I_k^- \left(\prod_{j=1}^n \prod_{a=1}^{E_j \cdot k} (v_j + az) \right). \end{aligned}$$

The twisted I -functions are almost I -functions for Y_+ and Y_- , except that they take values in the Chen–Ruan cohomology of X_{\pm} rather than Y_{\pm} . Their relation to the Gromov–Witten theory of Y_{\pm} comes via the so-called *quantum Lefschetz principle*, which can be phrased as follows. Let

$$(6.0.2) \quad \begin{aligned} I_Y^+(y, z) &:= e(E_+) I_{tw}^+(y, z) = \left(\prod_{j=1}^n v_j \right) I_{tw}^+(y, z) \\ I_Y^-(\tilde{y}, z) &:= e(E_-) I_{tw}^-(\tilde{y}, z) = \left(\prod_{j=1}^n v_j \right) I_{tw}^-(\tilde{y}, z). \end{aligned}$$

Theorem 6.1. ([12]) I_Y^{\pm} is the pushforward of an I -function on Y_{\pm} via the map $i_{\pm} : Y_{\pm} \rightarrow X_{\pm}$. More precisely, there exists a function $Z(y, z) \in H_{CR}^*(X_+)[z][[y]]$ and a map $y \mapsto \tau(y) \in H_{CR}^*(X_+)$ such that

$$I_Y^+(y, z) = (i_+)_* \left(z \mathbf{J}^{Y_+}(i_+^*(\tau(y)), z)(i_+^* Z(y, z)) \right).$$

The analogous statement holds for I_Y^- .

Proof. By Corollary 23 of [12], there exist $Z(y, z) \in H_{CR}^*(X_+)[z][[y]]$ and $y \mapsto \tau(y) \in H_{CR}^*(X_+)$ such that $I_{tw}^+(y, z)$ can be written as $z \mathbf{J}_{tw}^+(\tau(y), z)(Z(y, z))$, where \mathbf{J}_{tw}^+ is the E_+ -twisted J -function on X_+ (see [12] for a discussion). By Theorem 1.1 in [10],

$$i_+^* (z \mathbf{J}_{tw}^+(\tau(y), z)(Z(y, z))) = z \mathbf{J}^{Y_+}(i_+^*(\tau(y)), z)(i_+^* Z(y, z)).$$

The conclusion then follows since $(i_+)_* \circ i_+^*(\alpha) = e(E_+) \alpha$ for $\alpha \in H_{CR}^*(X_+)$. \square

As with the definition of ∂_j in Section 3.3, for each $1 \leq j \leq n$, we can define a differential operator $\bar{\partial}_j$ corresponding to E_j . Each E_j can be written as a linear combination $E_j = \sum_{i=1}^r b_j^i p_i^+ = \sum_{i=1}^r \tilde{b}_j^i p_i^-$. Define the operator

$$\bar{\partial}_j := \sum_{i=1}^r b_j^i y_i \frac{\partial}{\partial y_i} = \sum_{i=1}^r \tilde{b}_j^i \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}.$$

Then as in Section 3.3,

$$(6.0.3) \quad \bar{\partial}_j y^d = (E_j \cdot d) y^d;$$

$$(6.0.4) \quad \bar{\partial}_j e^{\sigma_{\pm}/z} = \frac{v_j}{z} e^{\sigma_{\pm}/z}.$$

We note that $E_j \in \overline{\mathbb{C}}_W$ for $1 \leq j \leq n$, thus $b_j^r = \tilde{b}_j^r = 0$.

As one might expect, we can express the terms of the E_\pm -twisted I -function in terms of certain differential operators applied to the terms of I^\pm . It is convenient to group the summands of the I -function in terms of powers of y_1, \dots, y_{r-1} . For $[k] \in (\mathbb{L} \otimes \mathbb{Q}) / \langle e \rangle$, define

$$G_{[k]}^+(y, z) := ze^{\sigma_+/z} \sum_{\substack{l \in \mathbb{K}_+ \\ l \equiv k \pmod{e}}} y^l I_l^+,$$

where I_l^+ is the l th coefficient, as in (3.2.2). Then

$$I^+(y, z) = \sum_{[k] \in \mathbb{K}_+ / \langle e \rangle} G_{[k]}^+(y, z).$$

We define $G_{[k]}^-(\tilde{y}, z)$ similarly. Note that $l \equiv k \pmod{e}$ if and only if $p_i^\pm \cdot l = p_i^\pm \cdot k$ for $1 \leq i \leq r-1$ (recall that $p_i^+ = p_i^-$ for $1 \leq i \leq r-1$). Thus $G_{[k]}^+(y, z)$ contains exactly those summands of $I^+(y, z)$ such that the exponent of y_i is $p_i^+ \cdot k$ for $1 \leq i \leq r-1$.

The linear transformation $L : H_{\text{CR}}^*(X_+)[z, z^{-1}] \rightarrow H_{\text{CR}}^*(X_-)[z, z^{-1}]$ preserves the power of y_i for $1 \leq i \leq r-1$. The change of variables from y to \tilde{y} maps \tilde{y}_i to $y_i \cdot y_r^{c_i}$ for some c_i . and the asymptotic expansion treats $\tilde{y}_1, \dots, \tilde{y}_{r-1}$ as constant. Define $\mathbb{G}_{[k]}(y, z) := L(G_{[k]}^-(\tilde{y}, z))$. By keeping track of powers of y_1, \dots, y_{r-1} , these observations allow us to refine the main result for toric varieties, Theorem 5.7, to the following:

Lemma 6.2. *For each $k \in \mathbb{L} \otimes \mathbb{Q}$, the linear transformation L of Theorem 5.7 maps $G_{[k]}^+(y, z)$ to the function $\mathbb{G}_{[k]}(y, z)$, which has asymptotic expansion $G_{[k]}^-(\tilde{y}, z)$ as $y_r \mapsto \infty$.*

Next we make the following observation, which follows immediately from (6.0.3) and (6.0.4) applied to the expression for I_Y^\pm .

$$(6.0.5) \quad \begin{aligned} I_Y^+(y, z) &:= \sum_{[k] \in \mathbb{K}_+ / \langle e \rangle} \prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az) G_{[k]}^+(y, z) \\ I_Y^-(\tilde{y}, z) &:= \sum_{[k] \in \mathbb{K}_- / \langle e \rangle} \prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az) G_{[k]}^-(\tilde{y}, z). \end{aligned}$$

Applying the operator $\prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az)$ to the relationship in Lemma 6.2, we conclude that the same relationship between the I -functions for X_+ and X_- holds for the (pushforwards of the) I -functions for Y_+ and Y_- .

Theorem 6.3. *The linear transformation $L : H_{\text{CR}}^*(X_+)[z, z^{-1}] \rightarrow H_{\text{CR}}^*(X_-)[z, z^{-1}]$ of Theorem 5.7 maps $I_Y^+(y, z)$ to a function $\mathbb{I}_Y(y, z)$ with asymptotic expansion $I_Y^-(\tilde{y}, z)$ as $y_r \mapsto \infty$.*

Proof. The operator $\prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az)$ commutes with the linear transformation L , as well as with the change of variables in (4.2.8) and the asymptotic expansion $y_r \mapsto \infty$. We conclude that

$$\begin{aligned} L \left(\prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az) G_{[k]}^+(y, z) \right) &= \prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az) G_{[k]}(y, z) \\ &\sim \prod_{j=1}^n \prod_{a=0}^{E_j \cdot k} (z\bar{\partial}_j + az) G_{[k]}^-(\tilde{y}, z). \end{aligned}$$

Summing over all $k \in \mathbb{L} \otimes \mathbb{Q}$, it follows that $L(I_Y^+(y, z)) \sim I_Y^+(y, z)$. \square

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